

# Pointed shape and global attractors for metrizable spaces

A. Giraldo<sup>a</sup> R. Jiménez<sup>b</sup> M. A. Morón<sup>c</sup> F.R.Ruiz del Portal<sup>c</sup>  
J. M. R. Sanjurjo<sup>c</sup>

<sup>a</sup>*Departamento de Matemática Aplicada, Facultad de Informática  
Universidad Politécnica de Madrid, Madrid 28040 Spain*

<sup>b</sup>*Instituto de Matemáticas UNAM, Cuernavaca, México*

<sup>c</sup>*Departamento de Geometría y Topología, Facultad de CC.Matemáticas  
Universidad Complutense de Madrid, Madrid 28040, Spain*

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## Abstract

In this paper we consider two notions of attractors for semidynamical systems defined in metric spaces. We show that Borsuk's weak and strong shape theories are a convenient framework to study the global properties which the attractor inherits from the phase space.

Moreover we obtain pointed equivalences (even in the absence of equilibria) which allow to consider also pointed invariants, like shape groups.

*Key words:* Shape theory, shape groups, (semi)dynamical system, attractor.

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## 1 Introduction

Several authors [15,17,18,20,22,26,37,38] have recently studied in various papers the global topological properties of attractors of dynamical systems. A common feature of these papers is the use of shape theory and Čech homology

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*Email addresses:* `agiraldo@fi.upm.es` (A. Giraldo),  
`rolando@matcuer.unam.mx` (R. Jiménez), `MA.Moron@Mat.UCM.Es` (M. A. Morón),  
`R.Portal@Mat.UCM.Es` (F.R.Ruiz del Portal), `Jose_Sanjurjo@Mat.UCM.Es` (J. M. R. Sanjurjo).

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and cohomology. The most general result in this direction can be found in [18], where the authors focus their attention on attractors for flows in Hausdorff topological spaces.

In all the cases, the shape being studied is unpointed, and it is a matter of considerable interest to know what the situation is in the pointed case. This problem can be described in the following terms: Suppose  $K$  is a global attractor of a flow in a metric space  $X$ . Suppose the attractor does not have equilibria. Is it true that the inclusion  $i : (K, x_0) \hookrightarrow (X, x_0)$  is a pointed shape equivalence for any choice of the point  $x_0 \in K$ ? If the attractor has equilibria, the difficulty is the same as in the unpointed case.

The interest of an affirmative answer lies in the fact that in such a case the pointed shape invariants, like the shape groups and homotopy pro-groups, are shared by the phase space and the attractor. The main idea, in order to find a solution to the problem, is to use Beck's Theorem [1] to replace the flow by another with equilibria in the attractor.

There are other shape-theoretical issues in the paper which are interesting even in the unpointed case. We show that the most natural shape theories to treat semidynamical systems in metrizable spaces are Borsuk's weak and strong shapes [7] (see also [4] and [5]). We also consider the most natural notion of attraction from the topological viewpoint i.e. the notion of compact attraction, which is more general than the notion of attraction of bounded sets considered by most of the authors in the literature of dynamical systems. This notion of bounded attraction is not only more restrictive, it is also non-topological, i.e., it depends on the metric considered in the phase space and it is not preserved by a change in the metric. Our conclusion is that global attractors with both notions of attraction have the same strong shape (in the sense of Borsuk) as the phase spaces.

The use of shape theory in the study of dynamical systems was initiated by Hastings in [24] and [25]. Other authors have shown how to apply shape theory to obtain global properties of attractors in the papers [3,13,15,17,22,37–39]. Shape theory and dynamical systems were also connected in [16]. On the other hand, shape theory was related to differential equations in [35] and it is the main tool used in [33] and [34] to define a Conley index [9,36] for discrete dynamical systems.

## 2 Basic notions about dynamical systems

Let  $X$  be a metric space. A dynamical (resp. semidynamical) system on  $X$  is a triad  $(X, \mathbb{R}, \pi)$  (resp.  $(X, \mathbb{R}_+, \pi)$ ) where  $\pi : X \times \mathbb{R} \rightarrow X$  (resp.  $\pi :$

$X \times \mathbb{R}_+ \longrightarrow X$ ) is a continuous map such that  $\pi(x, 0) = x$  for every  $x \in X$ , and  $\pi(\pi(x, t), s) = \pi(x, t + s)$  for every  $x \in X$  and every  $t, s \in \mathbb{R}$  (resp.  $t, s \in \mathbb{R}_+$ ). For every  $t$ , we will consider the map  $S_t : X \longrightarrow X$  given by  $S_t(x) = \pi(x, t)$ .

Given  $x \in X$ , the positive semi-orbit of  $x$  is  $\gamma^+(x) = \{S_t(x) \mid t \in \mathbb{R}_+\}$  and, if  $\{S_t\}$  is a dynamical system, the negative semi-orbit of  $x$  is defined as  $\gamma^-(x) = \{S_t(x) \mid t \in \mathbb{R}_-\}$ . The orbit of  $x$  is  $\gamma(x) = \gamma^+(x) \cup \gamma^-(x)$ . A set  $B \subset X$  is invariant if  $S_t(B) = B$ , for every  $t \in \mathbb{R}$  (for every  $t \in \mathbb{R}_+$  in the case of semidynamical systems).  $B$  is positively invariant if  $S_t(B) \subset B$ , for every  $t \in \mathbb{R}_+$ .

Given  $x \in X$  the positive limit ( $\omega$ -limit) of  $x$  is the set

$$\Lambda^+(x) = \{y \in X \mid \text{there exists } t_n \rightarrow \infty \text{ such that } S_{t_n}(x) \rightarrow y\} = \bigcap_{t \geq 0} \overline{\gamma^+(S_t(x))}.$$

The first prolongational positive limit of  $x$  is the set

$$J^+(x) = \{y \in X \mid \text{there exist } x_n \rightarrow x, t_n \rightarrow \infty \text{ such that } S_{t_n}(x_n) \rightarrow y\}.$$

Finally,

$$D^+(x) = \{y \in X \mid \text{there exist } x_n \rightarrow x, t_n \in \mathbb{R}_+ \text{ such that } S_{t_n}(x_n) \rightarrow y\},$$

(hence  $J^+(x) \subset D^+(x)$ ), and for any set  $C \subset X$ ,  $D^+(C) = \bigcup_{x \in C} D^+(x)$ .

A set  $K \subset X$  attracts a set  $C \subset X$  if for every  $\varepsilon > 0$  there exists  $T \in \mathbb{R}$  such that  $S_t(C) \subset B_\varepsilon(K)$  for every  $t \geq T$ .

$K$  is stable if for every open neighborhood  $V$  of  $K$  there is a neighborhood  $V_0$  of  $K$  such that  $S_t(V_0) \subset V$ , for every  $t \in \mathbb{R}_+$ .

**Definition 1** *Let  $\{S_t\}$  be a semidynamical system on a metric space  $X$ . A compact positively invariant set  $K \subset X$  is said to be a global attractor if it attracts all compact sets.*

The notion of attractor is related to that of a compact dissipative semigroup in [23, page 38].

We will also consider in this paper a stronger definition of global attractor. The definition will be analogous to Definition 1 replacing compact subsets by bounded subsets.

**Definition 2** *Let  $\{S_t\}$  be a semidynamical system on a metric space  $X$ . A*

compact invariant set,  $K \subset X$ , is said to be a global strong attractor if it attracts all bounded sets.

This definition has been used, associated to some classes of evolution equations, in [23], where compact and dissipative semigroups are studied. The study of global attractors in the literature is often related to some initial boundary-valued problems for partial differential equations.

Note that the notion of a global strong attractor is not topological, in the sense that global strong attraction is not preserved if we change the metric on  $X$  by another equivalent one. For example, if one considers the equation  $y' = -y$  in  $\mathbb{R}$ , with the usual metric  $d$ , then the origin is a global strong attractor but it is not a global strong attractor if we consider the equivalent metric  $d' = \frac{d}{1+d}$ .

Vaughan [40] proved that for a metrizable space  $X$ ,  $X$  is locally compact and separable if and only if there exists a metric  $d$  on  $X$  such that the subcompacts of  $X$  are exactly the closed and  $d$ -bounded subsets. Consequently the difference between both definitions of attractor is significant in the absence of local compactness of the phase space  $X$ .

We will need in the paper the following lemma. Some of the properties we obtain are given in [23, Chapter 3] but we include our proof for the sake of completeness. In order to simplify the notation we will write  $xt = S_t(x)$  and, for  $A \subset \mathbb{R}$ ,  $xA = \bigcup_{t \in A} S_t(x)$ .

**Lemma 3** *Let  $X$  be a metric space and let  $\{S_t : X \longrightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a global attractor  $K$ . Let  $C \subset X$  be a compactum. Then*

- i) *For every  $\varepsilon > 0$  there exist  $\delta > 0$  and  $T > 0$  such that  $B_\delta(C)[T, \infty) \subset B_\varepsilon(K)$ .*
- ii)  *$K$  is stable (see [23, Theorem 3.4.2])*
- iii)  *$D^+(C)$  is a subcompactum of  $C[0, \infty) \cup K$ .*
- iv) *For every open neighborhood  $V'$  of  $D^+(C)$  there is an open neighborhood  $V$  of  $C$  such that  $V[0, \infty) \subset V'$ .*

**PROOF.** i) Consider first  $C = \{x_0\}$ . Suppose there exist  $\{x_n\} \rightarrow x_0$  and  $\{t_n\} \rightarrow \infty$  with  $x_n t_n \notin B_\varepsilon(K)$ . Then, the set is a compact subset that is not attracted by  $K$ . This is a contradiction.

Suppose now that  $C \subset X$  is an arbitrary compactum and fix  $\varepsilon > 0$ . For each  $x \in C$  take the corresponding  $\delta_x$  and  $T_x$  as above. There is a finite subset  $\{x_1, x_2, \dots, x_r\} \subset C$  with  $C \subset \bigcup_{j \in \{1, 2, \dots, r\}} B_{\delta_{x_j}}(x_j)$ .

Now choose  $\delta$  such that  $B_\delta(C) \subset \cup_{j \in \{1,2,\dots,r\}} B_{\delta_{x_j}}(x_j)$  and  $T = \max_{j \in \{1,2,\dots,r\}} \{T_{x_j}\}$ . Then  $B_\delta(C)[T, \infty) \subset B_\varepsilon(K)$ .

ii) Consider  $V \subset X$  open such that  $K \subset V$ . Take  $\varepsilon > 0$  such that  $B_\varepsilon(K) \subset V$ . Take  $\delta < \varepsilon$  and  $T$  as in (i). Then  $B_\delta(K)[T, \infty) \subset B_\varepsilon(K) \subset V$ .

Using the continuity of  $\pi$  and the invariance of  $K$  there is  $\delta' > 0$  such that  $B_{\delta'}(K)[0, T] \subset B_\delta(K)$ . Then  $B_{\delta'}(K)[0, \infty) \subset V$ . Hence  $K$  is stable.

iii) Let  $\{y_k\} \subset D^+(C)$  be a sequence. Then, there exist  $\{x_n^k\} \subset C$ ,  $\{t_n^k\} \subset \mathbb{R}_+$  such that  $\{x_n^k t_n^k\} \rightarrow y_k$  and  $\{x_n^k\} \rightarrow x_k \in C$ .

We can also assume that  $\{x_k\} \rightarrow x_0 \in C$ . Inductively we can construct increasing sequences  $\{k_j\}$  and  $\{n_{k_j}\}$  such that  $d(x_{k_m}, x_0) < \frac{1}{m}$ ,  $d(x_{n_{k_m}}^{k_m} t_{n_{k_m}}^{k_m}, y_{k_m}) < \frac{1}{m}$  and  $d(x_{n_{k_m}}^{k_m}, x_{k_m}) < \frac{1}{m}$ .

Rewriting all above, we have two sequences  $\{x_n\} \rightarrow x_0$  and  $\{t_n\} \subset \mathbb{R}_+$  with  $d(x_n t_n, y_n) < \frac{1}{n}$ .

Assume first that  $\{t_n\}$  is bounded. Then we can suppose that  $\{t_n\} \rightarrow t_0$ . Consequently  $\{x_n t_n\} \rightarrow x_0 t_0 \in D^+(C)$ . Hence  $\{y_n\} \rightarrow x_0 t_0$ .

If  $\{t_n\}$  is unbounded, for any  $\varepsilon > 0$  there exists  $T_\varepsilon \in \mathbb{R}_+$  such that  $(\{x_n\} \cup \{x_0\})[T_\varepsilon, \infty) \subset B_\varepsilon(K)$ . This implies, taking subsequences if necessary, the existence of a sequence  $\{z_n\} \rightarrow z \in K$  with  $d(z_n, x_n t_n) < \frac{1}{n}$ . Then  $\{y_n\} \rightarrow z \in D^+(x_0) \subset D^+(C)$ .

iv) Suppose on the contrary that there exists  $\varepsilon > 0$  and  $\{x_n\} \rightarrow x_0 \in C$  such that  $d(x_n t_n, D^+(C)) \geq \varepsilon$ .

If  $\{t_n\}$  is bounded we can assume that  $\{x_n t_n\} \rightarrow x_0 t_0 \in D^+(C)$  which is contradictory.

On the other hand, if  $\{t_n\}$  is unbounded we can obtain, taking subsequences if necessary, a sequence  $\{z_n\} \subset K$  such that  $\{z_n\} \rightarrow z \in K$  and  $d(x_n t_n, z_n) < \frac{1}{n}$ . Then  $\{x_n t_n\} \rightarrow z \in D^+(C)$  which is again a contradiction.

**Remark 4** Note that (i) in the above lemma is not true if we only assume that  $K$  attracts points.

$K$  is said to be a *local attractor* if there exists a positively invariant neigh-

neighborhood  $W$  of  $K$  such that  $K$  attracts all compact subsets of  $W$ . The region of attraction of  $K$  is the maximal neighborhood with this property. It can be seen that, if  $K$  is a local attractor, then the region of attraction of  $K$  is precisely the set  $A(K) = \{x \in X \mid \emptyset \neq \Lambda^+(x) \subset K\}$ . As a consequence,  $A(K)$  is a positively invariant open subset of  $X$ .

We recommend [23] (also [2]) for more information on dynamical systems.

### 3 A Short Account of Shape Theory

We assume as known the basic facts and definitions in shape theory. In this section we recall several results and definitions which are less well known and that will be used in the sequel. The main reference is the book [7] by K.Borsuk.

Let  $X$  and  $Y$  be compacta, with  $Y$  contained in the Hilbert cube,  $Q$ . A sequence of continuous maps  $\{f_n : X \rightarrow Q \mid n \in \mathbb{N}\}$  such that for any open neighborhood  $V$  of  $Y$  in  $Q$  there exists  $n_0 \in \mathbb{N}$  such that  $f_n \overset{V}{\simeq} f_m$  for any  $n, m \geq n_0$ , is called an approximative sequence from  $X$  to  $Y$  [7, page 87]. The homotopy extension property of ANRs allows to extend  $\{f_n\}$  to a fundamental sequence (see [30, Lemma 1, page 333]).

We shall use in the sequel the following result.

**Theorem 5 ([7])** *Among countable compacta there are  $\aleph_1$  different shapes.*

There is an important part of the theory of shape for compacta concerning complement theorems. In these theorems the shape of compacta is determined by the topological type of its complements in some Euclidean space or in the Hilbert cube. An example of these results is the following theorem due to Geoghegan and Summerhill [14].

**Theorem 6** *Let  $A$  and  $B$  be two non-empty, strong  $Z_{n-k-2}$ -sets in  $E^n$ , where  $k \geq 0$  and  $n \geq 2k + 2$ . Then the following conditions are equivalent:*

- $Sh(A) = Sh(B)$ .
- The pointed spaces  $(E^n/A, |A|)$  and  $(E^n/B, |B|)$  are homeomorphic.
- $E^n/A$  is homeomorphic to  $E^n/B$ .

The classical shape theory used for arbitrary spaces is that developed in [30], [29], [11] and [31]. In the case of metrizable spaces there is a more geometrical description introduced earlier by Fox in [12].

In [7] (see also [4] and [5]) Borsuk extended his shape theory for compacta

to arbitrary metrizable spaces, obtaining a theory different from that of Fox. He essentially gave two extensions, the weak and the strong shape (the last should not be confused with what later was called strong shape, in the sense of coherent homotopy). None of these two extensions seems to have been specially appreciated by topologists. However, we hope to make clear in this paper that Borsuk's extensions are naturally associated with the study of attractors.

We now introduce Borsuk's notions of shape for arbitrary metric spaces. Let  $X$  and  $Y$  be two metric spaces. By Kuratowski-Wojdyslawski theorem we can suppose that  $X$  and  $Y$  are embedded as closed subsets in two AR-spaces  $P$  and  $Q$  respectively.

**Definition 7** *A weak fundamental sequence  $F = \{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  is a sequence of continuous maps  $\{f_n : P \rightarrow Q \mid n \in \mathbb{N}\}$  such that for any compact subset  $A \subset X$  there is a compact  $B \subset Y$  such that for any open neighborhood  $V$  of  $B$  in  $Q$  there are an open neighborhood  $U$  of  $A$  in  $P$  and  $n_0 \in \mathbb{N}$  such that  $f_n|_U \simeq f_m|_U : U \rightarrow V$  for any  $n, m \geq n_0$  ( $\simeq$  denotes the homotopy relation). In the above situation we say that the compact  $B$  is  $F$ -assigned to the compact  $A$ .*

*Two weak fundamental sequences  $F = \{f_n\}_{n \in \mathbb{N}}, G = \{g_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  are homotopic ( $F \stackrel{W}{\simeq} G$ ) if for any compact subset  $A \subset X$  there is a compact  $B \subset Y$  such that for any open neighborhood  $V$  of  $B$  in  $Q$  there are an open neighborhood  $U$  of  $A$  in  $P$  and  $n_0 \in \mathbb{N}$  such that  $f_n|_U \simeq (g_n)|_U : U \rightarrow V$  for any  $n \geq n_0$ .*

*A weak shape morphism from  $X$  to  $Y$  is an equivalence class of weak fundamental sequences  $F = \{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$ . Composition of weak shape morphisms is defined in the natural way. We obtain a category whose objects are the class of metric spaces and whose morphisms are the weak shape morphisms between them. We say that two metric spaces  $X$  and  $Y$  have the same weak shape ( $Sh_W(X) = Sh_W(Y)$ ) if they are isomorphic in this category.*

*We say that a metric space  $X$   $W$ -dominates a metric space  $Y$  ( $Sh_W(X) \geq Sh_W(Y)$ ) if there exist two weak fundamental sequences  $F = \{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  and  $G = \{g_n\}_{n \in \mathbb{N}} : Y \rightarrow X$  such that  $FG$  is homotopic to the identity.*

Not all sets have the weak shape of a compactum. For example:

**Theorem 8 ([7])** *If  $A$  is a countable compactum lying in  $E^n$ , then the set  $E^n \setminus A$  is not  $W$ -dominated by any compactum.*

**Definition 9** *A strong fundamental sequence  $F = \{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  is a weak fundamental sequence such that for any open neighborhood  $V'$  of  $Y$  in  $Q$  there are an open neighborhood  $U'$  of  $X$  in  $P$  and  $n_0 \in \mathbb{N}$  such that*

$f_n|_{U'} \simeq f_m|_{U'} : U' \rightarrow V'$  for any  $n, m \geq n_0$ .

Two strong fundamental sequences  $F = \{f_n\}_{n \in \mathbb{N}}, G = \{g_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  are homotopic ( $F \stackrel{S}{\simeq} G$ ) if they are homotopic as weak fundamental sequences and for any open neighborhood  $V'$  of  $Y$  in  $Q$  there are an open neighborhood  $U'$  of  $A$  in  $P$  and  $n_0 \in \mathbb{N}$  such that  $f_n|_{U'} \simeq g_n|_{U'} : U' \rightarrow V'$  for any  $n \geq n_0$ .

A strong shape morphism from  $X$  to  $Y$  is an equivalence class of strong fundamental sequences  $F = \{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$ . Analogously, we obtain in this way a category whose objects are the class of metric spaces and whose morphisms are the strong shape morphisms between them. We say that two metric spaces  $X$  and  $Y$  have the same strong shape ( $Sh_S(X) = Sh_S(Y)$ ) if they are isomorphic in this category.

We say that a metric space  $X$   $S$ -dominates a metric space  $Y$  ( $Sh_S(X) \geq Sh_S(Y)$ ) if there exist two strong fundamental sequences  $F = \{f_n\}_{n \in \mathbb{N}} : X \rightarrow Y$  and  $G = \{g_n\}_{n \in \mathbb{N}} : Y \rightarrow X$  such that  $FG$  is isomorphic to the identity.

**Remark 10**  $X$  homotopic to  $Y \Rightarrow Sh_S(X) = Sh_S(Y) \Rightarrow Sh_W(X) = Sh_W(Y)$ .

All the above notions and results have pointed versions.

## 4 Shape Theory as a tool for dynamics

This section is devoted to the study of several properties of attractors formulated in terms of Borsuk's (pointed) weak and strong shape theories.

### 4.1 The shape morphism induced by a semidynamical system

We start by showing how a semidynamical system with a global attractor induces shape morphisms in a natural way. Let  $X$  be a metric space and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a global attractor  $K$ . By Kuratowski-Wojdyslawski theorem we can suppose that  $X$  is embedded as a closed subset of an AR-space  $P$ . By Lemma 3, for each compact subspace  $C \subset X$  and for every neighborhood  $V$  of  $K$  in  $P$  there exists  $T > 0$  such that  $C[T, \infty) \subset V$ . This implies that the sequence of maps  $\{S_n : C \rightarrow P \mid n \in \mathbb{N}\}$  is an approximative map of  $C$  towards  $K$  that can be extended to a fundamental sequence and hence induces a shape morphism  $[S]_C : C \rightarrow K$ .



## 4.2 Attractors and components

The following result improves both Lemma 3.4.1 in [23] and Theorem 3.1 in [19].

**Proposition 11** *Let  $X$  be a metric space and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a global attractor  $K$ . Then, for every connected component  $X_\alpha$  of  $X$  there is exactly one component  $K_\alpha$  of  $K$  contained in  $X_\alpha$  and  $K_\alpha$  is a global attractor of the semiflow restricted to  $X_\alpha$ . In particular if  $X$  is connected then  $K$  is connected.*

**PROOF.** For every component  $X_\alpha$  of  $X$  take  $C_\alpha = X_\alpha \cap K$ . Since the semiorbit of every point  $x$  of  $X_\alpha$  is connected and  $K$  is compact then  $\emptyset \neq \Lambda^+(x) \subset C_\alpha$ .

To prove that  $C_\alpha$  is connected, suppose that  $C_\alpha = C_\alpha^1 \cup C_\alpha^2$  with  $C_\alpha^1$  and  $C_\alpha^2$  nonempty disjoint compacta. Consider  $X_\alpha^i = \{x \in X_\alpha \mid \Lambda^+(x) \subset C_\alpha^i\}$ . By its proper definition  $X_\alpha^1$  and  $X_\alpha^2$  are disjoint, and since  $X_\alpha^i \supset C_\alpha^i$  they are nonempty. We see now that  $X_\alpha = X_\alpha^1 \cup X_\alpha^2$ . Let  $x \in X_\alpha$ . Consider  $\varepsilon > 0$  such that  $B_\varepsilon(C_\alpha^1)$  and  $B_\varepsilon(C_\alpha^2)$  are disjoint. Then there exists  $T > 0$  such that  $x[T, \infty) \subset B_\varepsilon(C_\alpha^1) \cup B_\varepsilon(C_\alpha^2)$ . Since  $x[T, \infty)$  is connected, then  $x[T, \infty) \subset B_\varepsilon(C_\alpha^1)$  or  $x[T, \infty) \subset B_\varepsilon(C_\alpha^2)$ . Therefore  $\Lambda^+(x) \subset C_\alpha^1$  or  $\Lambda^+(x) \subset C_\alpha^2$ . We see finally that  $X_\alpha^1$  and  $X_\alpha^2$  are closed.

Consider a sequence  $\{y_m\} \subset X_\alpha^1$  such that  $y_m \rightarrow y$  and suppose that  $y \in X_\alpha^2$ . Take  $C = \{y\} \cup \{y_m : m \in \mathbb{N}\} \subset X_\alpha$  and consider the shape morphism  $[S]_C$  defined in 4.1. Consider also the induced map  $\Lambda_{[S]_C} : \square(C) \rightarrow \square(C_\alpha) = \square(C_\alpha^1) \cup \square(C_\alpha^2)$  between the corresponding spaces of components (see [7, page 214]). Since  $\{y_m\} \subset X_\alpha^1$  and  $y \in X_\alpha^2$ , then  $\Lambda_{[S]_C}(\{y_m\}) \subset \square(C_\alpha^1)$  and  $\Lambda_{[S]_C}(\{y\}) \subset \square(C_\alpha^2)$ . But this contradicts the continuity of  $\Lambda_{[S]_C}$ . Therefore  $X_\alpha^1$  and  $X_\alpha^2$  are closed, and hence  $X_\alpha$  is not connected. Since this is a contradiction,  $K_\alpha$  has to be connected.

**Proposition 12** *Let  $X$  be a metric space and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a global attractor  $K$ . Suppose that either  $X$  or  $K$  is locally connected. Then  $X$  and  $K$  have a finite number of components.*

**PROOF.** If  $X$  is locally connected, then its connected components are open sets. The former proposition implies that the components of  $K$  are also open in  $K$  and, since  $K$  is compact there can be only a finite number of them. Using again the former proposition we obtain that  $X$  also has a finite number of components.

On the other hand, if  $K$  is locally connected, since it is also compact it can only have a finite number of components. The above proposition implies that  $X$  also has a finite number of components.

### 4.3 Attractors and pointed shape

We have the following result for semiflows on ANRs.

**Theorem 13** *Let  $X$  be a metric ANR and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a global attractor  $K$ . Then, for every  $x_0 \in K$ , the inclusion  $i : (K, x_0) \rightarrow (X, x_0)$  induces a pointed weak shape equivalence, a pointed shape equivalence and a pointed strong shape equivalence.*

**Remark 14** *We note that, although the statement of Theorem 13 concerns pointed shape equivalences, the existence of an equilibrium is not required.*

To prove the assertion concerning shape and strong shape we need first to prove the following Lemmas.

**Lemma 15** *Let  $X$  be a connected metric ANR and let  $K$  be a subcontinuum of  $X$ . Let  $x_0 \in K$  and suppose that the inclusion  $i : (K, x_0) \hookrightarrow (X, x_0)$  induces a pointed weak shape equivalence. Then  $i$  induces pointed shape and strong shape equivalences.*

**PROOF.** Since  $(X, x_0)$  is a connected ANR, then it is pointed movable (in the sense of [7, page 168]). Since movability is a weak shape invariant, then  $(K, x_0)$  is pointed movable.

On the other hand, we have mentioned earlier that a pointed weak shape isomorphism induces, for every dimension, isomorphisms between the corresponding fundamental groups in the sense of Borsuk [7]. Since  $K$  is compact and  $X$  is an ANR, these fundamental groups agree with the standard shape groups. By a result of Keesling [27] (see also [32]), this implies that

$$i^* : \text{pro} - \pi_k(K, x_0) \longrightarrow \text{pro} - \pi_k(X, x_0) \simeq \pi_k(X, x_0)$$

is an isomorphism for every  $k \in \mathbb{N}$ . Here  $\text{pro} - \pi_k(K, x_0)$  is represented as the inverse sequence of groups

$$\cdots \longrightarrow \pi_k(U_{n+1}, x_0) \longrightarrow \pi_k(U_n, x_0) \longrightarrow \cdots \longrightarrow \pi_k(U_0, x_0),$$

where  $\{U_n\}$  is a neighborhood system on  $K$  in  $X$ . We shall also consider  $\text{pro} - \pi_*(K, x_0)$  as the inverse sequence of groups

$$\cdots \longrightarrow \prod_{j=0}^{\infty} \pi_j(U_{n+1}, x_0) \longrightarrow \prod_{j=0}^{\infty} \pi_j(U_n, x_0) \longrightarrow \cdots \longrightarrow \prod_{j=0}^{\infty} \pi_j(U_0, x_0)$$

and prove that

$$i^* : \text{pro} - \pi_*(K, x_0) \longrightarrow \text{pro} - \pi_*(X, x_0) \simeq \pi_*(X, x_0) = \prod_{j=0}^{\infty} \pi_j(X, x_0)$$

is an isomorphism.

In order to do it, we have to show that for every  $n_0 \in \mathbb{N}$  there exists  $n_1 \in \mathbb{N}$  and there exists  $\alpha_{n_1 n_0} : \prod_{j=0}^{\infty} \pi_j(X, x_0) \longrightarrow \prod_{j=0}^{\infty} \pi_j(U_{n_0}, x_0)$  such that the diagram

$$\begin{array}{ccc} \prod_{j=0}^{\infty} \pi_j(U_{n_0}, x_0) & \xleftarrow{i_{n_1 n_0}} & \prod_{j=0}^{\infty} \pi_j(U_{n_1}, x_0) \\ \downarrow j_{n_0} & \nwarrow \alpha_{n_1 n_0} & \downarrow j_{n_1} \\ \prod_{j=0}^{\infty} \pi_j(X, x_0) & \xleftarrow{\text{Id}} & \prod_{j=0}^{\infty} \pi_j(X, x_0) \end{array}$$

commutes, and this will be proved if we show that, given  $n_0$ , there exists  $n_1 \geq n_0$  such that for every  $k \in \mathbb{N}$  there exists  $\alpha_{n_1 n_0}^k : \pi_k(X, x_0) \longrightarrow \pi_k(U_{n_0}, x_0)$  such that the diagram

$$\begin{array}{ccc} \pi_k(U_{n_0}, x_0) & \xleftarrow{i_{n_1 n_0}} & \pi_k(U_{n_1}, x_0) \\ \downarrow j_{n_0} & \nwarrow \alpha_{n_1 n_0}^k & \downarrow j_{n_1} \\ \pi_k(X, x_0) & \xleftarrow{\text{Id}} & \pi_k(X, x_0) \end{array}$$

is commutative for every  $k$ .

Since  $(K, x_0)$  is movable, given  $n_0$ , there exists  $m(n_0)$  such that, for every  $n \geq m(n_0)$ , there exists  $r_n : (U_{m(n_0)}, x_0) \longrightarrow (U_n, x_0)$  such that

$$\begin{array}{ccc} (U_{m(n_0)}, x_0) & \xrightarrow{i_{m(n_0) n_0}} & (U_{n_0}, x_0) \\ & \searrow r_n & \nearrow i_{n n_0} \\ & (U_n, x_0) & \end{array}$$

is commutative.

On the other hand, since  $i : \text{pro-}\pi_k(K, x_0) \longrightarrow \text{pro-}\pi_k(X, x_0)$  is an isomorphism then, given  $n_0$ , for every  $k \in \mathbb{N}$ , there exists  $n_1^k \geq m(n_0)$  and there exists  $\alpha_{n_1^k n_0}^k : \pi_k(X, x_0) \longrightarrow \pi_k(U_{n_0}, x_0)$  such that

$$\begin{array}{ccc} \pi_k(U_{n_0}, x_0) & \xleftarrow{i_{n_1^k n_0}} & \pi_k(U_{n_1^k}, x_0) \\ j_{n_0} \downarrow & \swarrow \alpha_{n_1^k n_0}^k & \downarrow j_{n_1^k} \\ \pi_k(X, x_0) & \xleftarrow{\text{Id}} & \pi_k(X, x_0) \end{array}$$

is commutative for every  $k$ .

Then

$$\begin{array}{ccc} \pi_k(U_{n_0}, x_0) & \xleftarrow{i_{m(n_0)n_0}} & \pi_k(U_{m(n_0)}, x_0) \\ j_{n_0} \downarrow & \swarrow i_{n_1^k n_0} & \downarrow r_{n_1^k} \\ \pi_k(X, x_0) & \xleftarrow{\text{Id}} & \pi_k(X, x_0) \end{array}$$

$$\begin{array}{ccc} & & \pi_k(U_{n_1^k}, x_0) \\ & \swarrow \alpha_{n_1^k n_0}^k & \downarrow j_{n_1^k} \\ & & \pi_k(X, x_0) \end{array}$$

also commutes.

Finally, since

$$\begin{array}{ccccc} (U_{m(n_0)}, x_0) & \xrightarrow{r_{n_1^k}} & (U_{n_1^k}, x_0) & \xrightarrow{j_{n_1^k}} & (X, x_0) \\ & \searrow i_{m(n_0)n_0} & \downarrow i_{n_1^k n_0} & \nearrow j_{n_0} & \\ & & (U_{n_0}, x_0) & & \end{array}$$

commutes, we have that  $j_{n_1^k} r_{n_1^k} = j_{m(n_0)}$ , and therefore

$$\begin{array}{ccc} \pi_k(U_{n_0}, x_0) & \xleftarrow{i_{m(n_0)n_0}} & \pi_k(U_{m(n_0)}, x_0) \\ j_{n_0} \downarrow & \swarrow \alpha_{n_1^k n_0}^k & \downarrow j_{m(n_0)} \\ \pi_k(X, x_0) & \xleftarrow{\text{Id}} & \pi_k(X, x_0) \end{array}$$

Then, since  $(K, x_0)$  is movable and  $\text{pro-}\pi_*(K, x_0)$  is stable, by Theorem 7.6 in [10],  $(K, x_0)$  is a pointed FANR and, applying the Whitehead theorem in shape theory [10], we get that the inclusion is a shape equivalence.

Finally, the fact that the inclusion is a weak shape equivalence and a shape equivalence implies that it is also a strong shape equivalence.

**Lemma 16 (Keesling's reformulation [28] of Beck's Theorem [1])** *Let  $X$  be a metric space and let  $\{F_t : X \rightarrow X \mid t \in \mathbb{R}\}$  be a flow on  $X$  with  $S$  as set of fixed points. Then for any closed set  $S'$  containing  $S$  one can construct a new flow  $\{F'_t : X \rightarrow X \mid t \in \mathbb{R}\}$  with  $S'$  as set of fixed points. Moreover, for any  $x \in X \setminus S'$  with orbit  $O(x)$  under  $\{F_t\}$ , the orbit of  $x$  under  $\{F'_t\}$  is just the set of points which can be joined to  $x$  by an arc in  $O(x) \setminus S'$ .*

**Remark 17** *Keesling's reformulation of Beck's Theorem can be seen to hold also for semidynamical systems.*

**PROOF.** [Proof of Theorem 13] By the Kuratowski- Wojdyslawski theorem, we can assume that  $X$  is embedded as a closed subset of an AR-space  $P$ . Then, since  $X$  is an ANR, there is a retraction  $r : V \rightarrow X$  from a closed neighborhood  $V$  of  $X$  in  $P$ .

Take  $f_t : V \rightarrow X$  as  $f_t = S_t \circ r$  and consider  $F_t : P \rightarrow P$ , where  $F_t$  is any continuous extension of  $f_t$ . Let  $F = \{F_n\}_{n \in \mathbb{N}}$ .

Using (i) in Lemma 3,  $F$  is a weak fundamental sequence from  $X$  to  $K$ , and  $K$  is  $F$ -assigned to any compactum  $C \subset X$  (see definition in the introduction).

Consider  $i_{K,X} : K \rightarrow X$  and  $Id_K : K \rightarrow K$  the weak fundamental sequences induced by the inclusion and the identity respectively. Since  $F_n|_K$  is homotopic in  $K$  to the identity in  $K$ , for every  $n \in \mathbb{N}$ , then  $F \circ i_{K,X} \xrightarrow{W} Id_K$ .

In order to prove that  $i_{K,X} \circ F \xrightarrow{W} Id_X$  note first that for any compact  $C \subset X$ ,  $D^+(C)$  is also compact ((iii) in Lemma 3). Let  $A' \subset V$  be an open subset of  $P$  containing  $D^+(C)$  and let  $V' = A' \cap X$ . Applying (iv) in Lemma 3 we have that there is a neighborhood  $V_0$  of  $C$  in  $X$  with  $S_t(V_0) = F_t(V_0) = f_t(V_0) \subset V' \subset A'$  for  $t \in \mathbb{R}_+$ .

Take  $U = r^{-1}(V_0) \cap A'$ , then  $U$  is a neighborhood of  $C$  in  $P$ . Consider the continuous map  $H : U \times [0, \infty) \rightarrow V' \subset A'$  defined by  $H(x, t) = S_t(r(x))$ . Then  $H$  induces homotopies between the identity and every  $S_t$  on  $C$  inside  $V'$ . Using now the fact that  $V'$  is an ANR we have that there is an open neighborhood  $U'$  of  $C$  contained in  $U$  such that  $i|_{U'}$  and  $r|_{U'}$  are homotopic in  $V'$  (where  $i$  is the identity). Then  $F_n|_{U'} \simeq i|_{U'}$  in  $V'$  for every  $n \in \mathbb{N}$  and therefore  $i : K \rightarrow X$  induces a weak shape equivalence.

To prove that  $i$  induces a pointed weak shape equivalence we use Keesling's reformulation of Beck's Theorem to get a new semiflow  $\{S'_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$

having  $x_0$  as an equilibrium. We see that  $K$  is also a global attractor for  $\{S'_t\}$ . Suppose it is not. Then there exists a positively invariant neighborhood  $U$  of  $K$  and there exist  $\{y_n\} \subset X$ ,  $y_n \rightarrow y_0$ , and  $\{t_n\} \subset \mathbb{R}_+$ ,  $t_n \rightarrow \infty$ , such that  $S'_{t_n}(y_n) \notin U$  for every  $n \in \mathbb{N}$ . On the other hand, there exists  $t_0 \in \mathbb{R}_+$  such that  $S'_{t_0}(y_0) \in U$ , and, by the continuity of the semiflow  $\{S'_t\}$ , there exists  $n_0 \in \mathbb{N}$  such that  $S'_{t_0}(y_n) \in U$  for every  $n \geq n_0$ . But then  $S'_t(y_n) \in U$  for every  $n \geq n_0$  and every  $t \geq t_0$  and this is a contradiction.

Finally, if  $X$  is connected, we apply Lemma 15 to get that  $i$  induces a shape and a strong shape equivalence. If  $X$  is not connected, the result follows from the same argument applied to each component.

**Corollary 18** *Let  $X$  be a metric ANR and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a global attractor  $K$ . Then the Čech homology and cohomology groups, the shape groups, and the homotopy, homology and cohomology pro-groups of  $X$  and  $K$  are isomorphic.*

**Corollary 19** *Let  $X$  be a metric ANR and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system with a local attractor  $K$ . Then, for every  $x_0 \in K$ , the inclusion  $i : (K, x_0) \rightarrow (A(K), x_0)$  induces a pointed weak shape equivalence, a shape equivalence and a strong shape equivalence.*

The following corollary extends to this context a result of [8].

**Corollary 20** *Let  $X$  be a metric ANR and let  $\{S_t : X \rightarrow X \mid t \in \mathbb{R}_+\}$  be a semidynamical system. Assume that  $K \in \text{ANR}$  is a local attractor. Then  $K$  and  $A(K)$  have the same homotopy type and, as a consequence, their Euler characteristics  $\mathcal{X}(K)$  and  $\mathcal{X}(A(K))$  agree.*

In view of Theorem 13 and Proposition 12 every ANR which admits a semidynamical system with a global attractor must have a finite number of components. The question can be naturally raised whether every ANR with a finite number of components admits a semidynamical system with a global attractor. The following corollary shows that the answer is very far from positive.

**Corollary 21** *For every  $n \geq 2$  there is a family  $\{G_i\}_{i \in I}$  of open connected dense subsets of  $\mathbb{R}^n$ , with  $\text{Card}(I) = \aleph_1$ , with the following properties:*

- a)  $G_i$  and  $G_j$  are not homeomorphic if  $i \neq j$ .
- b) If  $\{S_t : G_i \rightarrow G_i\}$  is a semidynamical system then it does not admit global attractors.

**PROOF.** By Theorem 5, there are  $\aleph_1$  countable compact metric spaces of different shapes. We can place a copy  $F_i$  of each of them in  $\mathbb{R}^n$  as a strong

$Z_{n-2}$ -set. Consider  $G_i = \mathbb{R}^n \setminus F_i$ . Then, using Theorem 6 we obtain a).

In order to prove b) it is enough to apply Theorems 8 and 13.

#### 4.4 Attractors and strong shape

For strong attractors, in the sense of Definition 2, the most significant result is the following:

**Theorem 22** *Let  $X$  be a metric space. Suppose that  $\{S_t : X \longrightarrow X\}$  is a semidynamical system with a global strong attractor  $K$ . Assume  $B \subset X$  is a bounded positively invariant neighborhood of  $K$ . Then, the inclusion  $i : K \rightarrow B$  induces a strong shape equivalence. In particular,  $Sh_S(K) = Sh_S(B)$ . Moreover, if  $\{S_t : X \longrightarrow X\}$  is a dynamical system, then  $i : (K, x_0) \rightarrow (B, x_0)$  is a pointed strong shape equivalence, for every  $x_0 \in K$ .*

**PROOF.** Assume that  $B$  is embedded as a closed subset of an AR-space  $P$ . Since  $K$  attracts  $B$ , we can obtain a base  $\{V_n\}_{n \in \mathbb{N}}$  of open neighborhoods of  $K$  in  $P$  with the following properties:

- 1)  $S_m(B) \subset V_n$  for  $m \geq n$ .
- 2)  $S_n|_B$  and  $S_m|_B$  are homotopic in  $V_n$  for  $m \geq n$ .

Now, following the procedure (essentially based on the homotopy extension property) described in Lemma 3 in [30, page 333], we can construct a sequence  $\{f_n : P \rightarrow P \mid n \in \mathbb{N}\}$  of maps and a sequence of closed neighborhoods  $\{U_n\}_{n \in \mathbb{N}}$  of  $B$  in  $P$  such that  $f_n|_B = S_n$  and  $f_m|_{U_n}$  and  $f_n|_{U_n}$  are homotopic in  $V_n$  for  $m \geq n$ . Consequently, if we use the  $F$ -assignment as in the proof of Theorem 13 and the above property we have that  $F = \{f_n\}_{n \in \mathbb{N}} : B \rightarrow K$  is a strong fundamental sequence.

Consider  $i_{K,B} : K \rightarrow B$  and  $Id_K : K \rightarrow K$  the weak fundamental sequences induced by the inclusion and the identity respectively. Since  $f_n|_K$  is homotopic in  $K$  to the identity in  $K$ , for every  $n \in \mathbb{N}$ , then  $F \circ i_{K,B} \simeq Id_K$ .

In order to prove that  $i_{K,B} \circ F \stackrel{S}{\simeq} Id_B$  observe that for every neighborhood  $V$  of  $B$  in  $P$  there exists  $n_0 \in \mathbb{N}$  such that, for  $n \geq n_0$ ,  $V_n \subset V$ , and hence  $f_n|_{U_{n_0}}$  and  $f_{n_0}|_{U_{n_0}}$  are homotopic in  $V$ . On the other hand, since  $B$  is invariant,  $f_0|_B$  and  $f_{n_0}|_B$  are homotopic in  $B$  and hence there exists a neighborhood  $U \subset U_{n_0}$  of  $B$  in  $P$  such that  $f_0|_U$  and  $f_{n_0}|_U$  are homotopic in  $V$ . Therefore, for every  $n \geq n_0$ ,  $f_n|_U$  is homotopic in  $V$  to the identity.

Consider now a compact set  $C \subset B$ . Then  $D^+(C) \cup K$  is also compact. Let  $V$  be an open neighborhood of  $D^+(C) \cup K$  in  $P$ . In a similar way as before, there exists  $n_0 \in \mathbb{N}$  such that, for  $n \geq n_0$ ,  $V_n \subset V$ , and hence  $f_n|_{U_{n_0}}$  and  $f_{n_0}|_{U_{n_0}}$  are homotopic in  $V$ . On the other hand, by Lemma 3, there exists  $U$  closed neighborhood of  $C$  in  $B$  such that  $U[0, \infty) \subset V$ . This implies that  $f_0|_U$  and  $f_{n_0}|_U$  are homotopic in  $V$ . Then there exists  $U' \subset U_{n_0}$  open subset of  $P$  such that  $U \subset U'$ , and  $f_0|_{U'}$  and  $f_{n_0}|_{U'}$  are homotopic in  $V$ . Therefore, for every  $n \geq n_0$ ,  $f_n|_{U'}$  is homotopic in  $V$  to the identity and, hence,  $i : K \rightarrow B$  is a strong shape equivalence.

Now, a careful analysis of the proof of Beck's theorem shows that when  $\{S_t : X \rightarrow X\}$  is a dynamical system we can conveniently alter the flow so that  $K$  is a global strong attractor with an equilibrium  $x_0$  for any choice of  $x_0 \in K$ . The rest of the proof is similar to that of Theorem 13.

Using the standard technique of primitive Lyapunov functions, we can get a strong deformation retraction from  $X$  to a bounded positively invariant neighborhood of  $K$ ,  $B$ . Then, applying Corollary 11.4 in [7, page 118] we obtain the following corollary of theorem 22.

**Corollary 23** *Let  $X$  be an arbitrary metric space and  $\{S_t\}_{t \in \mathbb{R}_+}$  a semidynamical system on  $X$  with a global strong attractor  $K$ . Then, the inclusion  $i : K \rightarrow X$  induces a strong shape equivalence and, in particular,  $Sh_S(K) = Sh_S(X)$ . Moreover, if  $\{S_t\}_{t \in \mathbb{R}_+}$  is a dynamical system, we have that  $i : (K, x_0) \rightarrow (X, x_0)$  is a pointed strong shape equivalence for any choice of the point  $x_0 \in K$ .*

*As a consequence,  $K$  and  $X$  share all strong shape invariants. In particular  $Sh(K) = Sh(X)$ , their Čech homology and cohomology groups agree and  $K$  is connected if and only if  $X$  is connected. When  $\{S_t\}_{t \in \mathbb{R}_+}$  is a dynamical system,  $K$  and  $X$  also have isomorphic shape and homotopy pro-groups.*

The fact that  $Sh_S(K) = Sh_S(X)$  implies that  $Sh(K) = Sh(X)$  is proved in [21], where it is also shown that the converse result is not true.

#### 4.5 The shape of positively invariant regions

We prove in this section that closed positively invariant regions of the phase space  $X$  have the shape of subcompacta of the global attractor.

**Theorem 24** *Let  $X$  be an arbitrary metric space and  $\{S_t\}_{t \in \mathbb{R}_+}$  a semidynamical system on  $X$  with a global attractor  $K$ . Let  $L$  be a closed positively invariant region in  $X$ . Then  $L$  has the shape of a positively invariant subcompactum of  $K$ .*



**PROOF.** Consider  $K' = L \cap K$ . Then  $K'$  is a global attractor of the flow  $\{S'_t\}_{t \in \mathbb{R}_+} = \{S_t|_L\}_{t \in \mathbb{R}_+}$  and, as a consequence,  $Sh(L) = Sh(K')$ , where  $K'$  is a positively invariant subcompactum of  $K$ .

**Corollary 25** *If  $\dim(K) = n$  then every positively invariant closed set has the shape of a  $m$ -dimensional compactum ( $m \leq n$ ). As a consequence its Čech homology and cohomology groups vanish for  $m > n$ .*

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